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N-Particle Correlations in the McKean Model

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In the McKean model the BBGKY hierarchy is equivalent to a simple hierarchy of coupled equations for the p-particle correlation functions. Approximate solutions are obtained by truncating the hierarchy. The convergence of the truncation method is studied by comparison with the exact solution for the model, which can be given in closed form. In the long-time limit the exact solution is linearized around the equilibrium value, showing the decay of the correlations. It turns out that p-particle correlations decay p times faster than the non-equilibrium one-particle distribution.

KEY WORDS: BBGKY hierarchy; correlations; truncation; relaxation times.

1. INTRODUCTION

The time evolution of a many-particle system is completely described by the BBGKY hierarchy for the reduced distribution functions.[1] To calculate observables of the system, one must truncate the hierarchy somewhere to close the set of equations. This is done in most cases after the first equation, because coupled nonlinear integrodifferential equations are hard to handle. If the p-particle distribution function is represented in clusters of one-particle distribution functions and correlation functions, truncation after the first equation means neglecting the two-particle correlations, as in the famous Boltzmann equation. The quality of this approximation can be tested in most cases only empirically.

It is assumed, however,[2] that for a description of a many-particle system near the equilibrium ($t \to \infty$) the correlations are no longer impor-

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tant, because they decay much faster than the one-particle distribution function tends to the equilibrium. Therefore, it seems important to test this assumption in simple models where also the higher hierarchy equations can be solved. The McKean model, briefly described in Section 2, is one of the simplest, nontrivial models of a many-particle system where questions such as entropy production and the evolution of molecular chaos have already been studied.[3]

In this paper we go far beyond the assumption of molecular chaos. In Section 3, we obtain the BBGKY hierarchy of the McKean model and describe the reduced distribution functions in a cluster representation.[1] In the limit of large particle numbers these equations are transformed in Section 4 into a hierarchy for the correlation functions. It is also possible to obtain an exact solution for the reduced distribution functions in this limit[4-6]; this is reviewed in Section 5. The fixed points of the coupled equations for the evolution of the one-particle distribution function f_+ and the p-particle correlation functions g_p are considered in Section 6. The effects of truncation on the pth level and the convergence toward the exact solution are studied numerically. Near the equilibrium fixed point the exact solution can be linearized as done in Section 7, showing the decay of the correlation functions. It turns out that the p-particle correlations decay p times faster than the nonequilibrium one-particle distribution. This confirms the assumption that higher correlations become increasingly unimportant when a system approaches equilibrium. The conclusions of the paper are summed up in Section 8 and some details of the calculations are presented in the appendices.

2. THE MCKEAN MODEL

McKean[3,7,8] considered a system of N particles that can move with a constant velocity $e_i = \pm 1$. The probability of the interaction between two particles in the time interval dt is

$$P = (2/N)\,dt$$

If the interacting particles i and j have the velocities e_i and e_j initially, they will have the velocities e_i^* and e_j^* after the interaction with equal probability $\frac{1}{2}$:

$$\left.\begin{aligned} e_i^* &= e_i \\ e_j^* &= e_i e_j \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} e_i^* &= e_i e_j \\ e_j^* &= e_j \end{aligned}\right.$$

or, more explicitly,

$$
\begin{array}{lll}
 & \nearrow & +1,\ +1 \\
+1,\ +1 & & \\
 & \searrow & +1,\ +1 \\
 & \nearrow & -1,\ -1 \\
-1,\ +1 & & \\
 & \searrow & -1,\ +1
\end{array}
\qquad
\begin{array}{lll}
 & \nearrow & +1,\ -1 \\
+1,\ -1 & & \\
 & \searrow & -1,\ -1 \\
 & \nearrow & -1,\ +1 \\
-1,\ -1 & & \\
 & \searrow & +1,\ -1
\end{array}
$$

Two particles having initial velocities $+1$ will remain in this state after the interaction. The situation is completely different if the two particles move with velocities -1. Therefore, the interaction is not invariant with respect to time inversion. Let $\rho(e_1,..., e_N; t)$ be the probability of the N particles having velocities $e_1,..., e_N$ at time t. This function is called the N-particle distribution function and it is normalized according to

$$\sum_{e_1,...,e_N=\pm 1} \rho(e_1,..., e_N; t) = 1 \tag{2.1}$$

We restrict ourselves to distribution functions that are symmetric in all variables.

The reduced p-particle distribution functions are defined according to

$$f_p(e_1,..., e_p; t) = \sum_{e_{p+1},...,e_N=\pm 1} \rho(e_1,..., e_N; t) \qquad (1 \leqslant p \leqslant N) \tag{2.2}$$

With the interaction defined above, we obtain a kinetic equation for the N-particle distribution function

$$\begin{aligned}
\frac{\partial}{\partial t} \rho(e_1,..., e_N; t)& \\
= \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} &[\rho(e_1,..., e_i,..., e_i e_j,..., e_N; t) \\
&+ \rho(e_1,..., e_i e_j,..., e_j,..., e_N; t) \\
&- 2\rho(e_1,..., e_i,..., e_j,..., e_N; t)]
\end{aligned} \tag{2.3}$$

It can immediately be seen that this equation conserves the normalization (2.1). A kinetic equation for the reduced one-particle distribution function f_1 can be derived from (2.3) and (2.2) by summation of (2.3) over velocities $e_2,..., e_N$:

$$\begin{aligned}
\frac{\partial f_1(e_1; t)}{\partial t} = \frac{N-1}{N} \sum_{e_2=\pm 1} &[f_2(e_1, e_1 e_2; t) + f_2(e_1 e_2, e_2; t) \\
&- 2f_2(e_1, e_2; t)]
\end{aligned} \tag{2.4}$$

We decompose f_2 into a part that factorizes into a product of two one-particle distribution functions and a remainder, which is the (irreducible) two-particle correlation[1]:

$$f_2(e_1, e_2; t) = f_1(e_1; t) f_1(e_2; t) + g_2(e_1, e_2; t) \tag{2.5}$$

To close Eq. (2.4), we neglect the two-particle correlations completely:

$$g_2(e_1, e_2; t) = 0 \tag{2.6}$$

Taking also the limit $N \to \infty$ in (2.4), we then obtain the Boltzmann equation of the McKean model:

$$\begin{aligned} \frac{\partial f_1(e_1; t)}{\partial t} &= f_1(e_1; t)[f_1(e_1; t) + f_1(-e_1; t)] \\ &\quad + [f_1(e_1; t) f_+ + f_1(-e_1; t) f_-] - 2f_1(e_1; t) \end{aligned} \tag{2.7}$$

where

$$f_+ = f_1(e_1 = 1; t) \qquad \text{and} \qquad f_- = f_1(e_1 = -1; t) \tag{2.8}$$

with the normalization condition

$$f_+ + f_- = 1 \tag{2.9}$$

These approximations are of course equivalent to the assumption of molecular chaos:

$$\lim_{N \to \infty} f_2(e_1, e_2; t) = [\lim_{N \to \infty} f_1(e_1; t)][\lim_{N \to \infty} f_1(e_2; t)]$$

If we choose $e_1 = 1$ in (2.7) and use (2.8) and (2.9), we obtain an equation for f_+:

$$\partial f_+(t)/\partial t = [2f_+(t) - 1][f_+(t) - 1] \tag{2.10}$$

This equation can be solved[3,7]

$$f_+(t) = \frac{1}{2}\left[1 + \frac{Ae^{-t}}{1 - A(1 - e^{-t})}\right] \tag{2.11}$$

with

$$A = 2f_+(0) - 1$$

For $t \to \infty$, we obtain

$$f_+(\infty) = \frac{1}{2} \qquad \text{if} \quad f_+(0) \neq 1 \tag{2.12}$$

and

$$f_+(\infty)=f_+(t)=1 \qquad \text{if} \quad f_+(0)=1 \tag{2.13}$$

In fact, the two possible values of $f_+(\infty)$ are the two fixed points of Eq. (2.10), since the equation

$$[2f_+(t)-1][f_+(t)-1]=0$$

is satisfied by

$$f_+(t)=\frac{1}{2} \qquad \text{and} \quad f_+(t)=1 \tag{2.14}$$

In order to decide whether the fixed points are attractive or repulsive, we linearize (2.10) about the values $\frac{1}{2}$ and 1. We obtain

$$\begin{aligned} \dot{\varepsilon}(t) &= -\varepsilon(t) && \text{for} \quad \varepsilon(t)=f_+(t)-\frac{1}{2} \\ \dot{\varepsilon}(t) &= \varepsilon(t) && \text{for} \quad \varepsilon(t)=f_+(t)-1 \end{aligned} \tag{2.15}$$

where $\varepsilon(t)$ is infinitesimally small. The positive eigenvalue $+1$ corresponds to the repulsive fixed point $f_+(t)=1$, which is therefore unstable, while the negative eigenvalue -1 corresponds to the attractive fixed point $f_+=\frac{1}{2}$, which is the equilibrium point of the system.

3. HIERARCHY EQUATIONS FOR THE REDUCED DISTRIBUTION FUNCTIONS

To go beyond approximation (2.6), we must derive equations for higher reduced distribution functions. In general, the equation for the reduced p-particle distribution function is obtained by summation of Eq. (2.3) over velocities $e_{p+1},\ldots, e_N$. This leads to a coupling to the reduced $(p+1)$-particle distribution function. In this way, we obtain the BBGKY hierarchy equations of the McKean model.

The first three equations of the hierarchy are

$$\begin{aligned} \frac{\partial f_1(e_1)}{\partial t}=\frac{N-1}{N}\sum_{e_2=\pm 1} &[f_2(e_1, e_1 e_2) \\ &+f_2(e_1 e_2, e_2)-2f_2(e_1, e_2)] \end{aligned} \tag{3.1a}$$

$$\frac{\partial f_2(e_1, e_2)}{\partial t} = \frac{1}{N} [f_2(e_1, e_1 e_2) + f_2(e_1 e_2, e_2)]$$

$$+ \frac{N-2}{N} \sum_{e_3 = \pm 1} [f_3(e_1, e_2, e_1 e_3)$$

$$+ f_3(e_1 e_3, e_2, e_3) + f_3(e_1, e_2, e_2 e_3)$$

$$+ f_3(e_1, e_2 e_3, e_3) - 2 \frac{2N-3}{N-2} f_3(e_1, e_2, e_3)] \tag{3.1b}$$

$$\frac{\partial f_3(e_1, e_2, e_3)}{\partial t} = \frac{1}{N} [f_3(e_1, e_1 e_2, e_3) + f_3(e_1 e_2, e_2, e_3)$$

$$+ f_3(e_1, e_2, e_1 e_3) + f_3(e_1 e_3, e_2, e_3)$$

$$+ f_3(e_1, e_2, e_2 e_3) + f_3(e_1, e_2 e_3, e_3)]$$

$$+ \frac{N-3}{N} \sum_{e_4 = \pm 1} \Big[f_4(e_1, e_2, e_3, e_1 e_4)$$

$$+ f_4(e_1, e_2, e_3, e_2 e_4) + f_4(e_1, e_2, e_3, e_3 e_4)$$

$$+ f_4(e_1 e_4, e_2, e_3, e_4) + f_4(e_1, e_2 e_4, e_3, e_4)$$

$$+ f_4(e_1, e_2, e_3 e_4, e_4) - 6 \frac{N-2}{N-3} f_4(e_1, e_2, e_3, e_4) \Big] \tag{3.1c}$$

The time dependence of the f_ps has been suppressed for simplicity. In the limit $N \to \infty$ the BBGKY hierarchy converges to the Boltzmann hierarchy.[4-6] Taking this limit, we obtain from Eqs. (3.1)

$$\frac{\partial}{\partial t} f_p(e_1,..., e_p; t)$$

$$= \sum_{i=1}^{p} \sum_{e_{p+1} = \pm 1} [f_{p+1}(e_1,..., e_i,..., e_i e_{p+1}; t)$$

$$+ f_{p+1}(e_1,..., e_i e_{p+1},..., e_{p+1}; t) - 2p f_{p+1}(e_1,..., e_{p+1}; t)] \tag{3.2}$$

The usual way to handle a many-particle problem is to decompose the reduced distribution functions into their irreducible parts and to calculate the time evolution of these functions. Therefore, we use the cluster representation:

$$f_2(e_1, e_2) = f_1(e_1) f_1(e_2) + g_2(e_1, e_2) \tag{3.3a}$$

$$f_3(e_1, e_2, e_3) = f_1(e_1) f_1(e_2) f_1(e_3) + f_1(e_1) g_2(e_2, e_3) + f_1(e_2) g_2(e_1, e_3) + f_1(e_3) g_2(e_1, e_2) + g_3(e_1, e_2, e_3) \tag{3.3b}$$

where g_2 and g_3 are the two- and three-particle correlation functions.

Because of Eq. (2.2), the p-particle correlation function must satisfy the condition

$$\sum_{e_k = \pm 1} g_p(e_1, \ldots, e_p) = 0, \qquad 1 \leqslant k \leqslant p \tag{3.4}$$

Since any e_k can only take the values ± 1, we obtain the relation

$$g_p(\{e_p\} = \{+1\}) = (-1)^j g_p(\{e_{p-j}\} = \{+1\}, \{e_j\} = \{-1\}), \qquad j \leqslant p \tag{3.5}$$

where $\{e_p\}$ stands for the p-tuple $e_1, \ldots, e_p$.

This result reduces enormously the number of quantities that must be calculated in order to solve the many-particle problem. It means that the absolute value of the p-particle correlation functions is independent of the velocity direction of the various particles. Therefore, we only need to deal with the hierarchy equation for the p-particle distribution function where all p velocities are $+1$. This equation contains all the independent quantities that are necessary to calculate the various reduced distribution functions.

We obtain from Eq. (3.2)

$$\frac{\partial}{\partial t} f_p(\{e_p\} = \{+1\}) = p[2f_{p+1}(\{e_{p+1}\} = \{+1\}) + f_{p+1}(\{e_p\} = \{+1\}, e_{p+1} = -1) + f_{p+1}(\{e_{p-1}\} = \{+1\}, e_p = e_{p+1} = -1) - 2f_p(\{e_p\} = \{+1\})] \tag{3.6}$$

The reduced distribution functions f_p occurring here can be expressed in the cluster representation analogous to Eqs. (3.3):

$$f_p(\{e_p\} = \{+1\}) = f_+^p + \sum_{i=1}^{p-2} \binom{p}{i} f_+^i \, g_{p-i} + g_p \tag{3.7a}$$

$$f_p(\{e_{p-1}\} = \{+1\}, e_p = -1) = f_+^{p-1}(1 - f_+) - \sum_{i=1}^{p-2} \binom{p}{i} f_+^i \, g_{p-i} + \sum_{i=1}^{p-2} \binom{p-1}{p-i} f_+^{i-1} g_{p-i} + g_p \tag{3.7b}$$

$$f_p(\{e_{p-2}\}=\{+1\}, e_{p-1}=e_p=-1)$$
$$= f_+^{p-2}(1-f_+)^2 + \sum_{i=1}^{p-2} \binom{p}{i} f_+^i g_{p-i}$$
$$-2\sum_{i=1}^{p-2} \binom{p-1}{p-i} f_+^{i-1} g_{p-i} + \sum_{i=2}^{p-2} \binom{p-2}{p-i} f_+^{i-2} g_{p-i} + g_p \qquad (3.7c)$$

In this way, every reduced distribution function can be expressed in terms of f_+ and g_j with $j \leqslant p$.

Insertion of Eqs. (3.7) in (3.6) leads to

$$\frac{\partial}{\partial t}\left[f_+^p + \sum_{i=1}^{p-2} \binom{p}{i} f_+^i g_{p-i} + g_p\right]$$
$$= p(2f_+^{p+1} - 3f_+^p + f_+^{p-1})$$
$$+ \sum_{i=1}^{p-1} 2p \binom{p+1}{i} f_+^i g_{p-i+1} - \sum_{i=1}^{p-1} p \binom{p}{i-1} f_+^{i-1} g_{p-i+1}$$
$$+ \sum_{i=2}^{p-1} p \binom{p-1}{i-1} f_+^{i-2} g_{p-i+1} - \sum_{i=1}^{p-2} 2p \binom{p}{i} f_+^i g_{p-i} - 2p(g_p - g_{p+1})$$
$$(3.8)$$

As shown in the next section, this can be regarded as an equation coupling the p-particle correlation to the $(p \pm 1)$-particle correlations.

4. HIERARCHY EQUATIONS FOR THE p-PARTICLE CORRELATION FUNCTION

To demonstrate how Eq. (3.8) can be solved for the correlation functions, we deal with the first two hierarchy equations of (3.8):

$$p=1: \qquad \frac{\partial}{\partial t} f_+ = 2f_+^2 - 3f_+ + 1 + 2g_2 \qquad (4.1)$$

$$p=2: \qquad \frac{\partial}{\partial t} f_+^2 + g_2 = 4f_+^3 - 6f_+^2 + 2f_+ + 12f_+ g_2 - 6g_2 + 4g_3 \qquad (4.2)$$

Multiplication of (4.1) with $2f_+$ and insertion in (4.2) leads to

$$\frac{\partial}{\partial t} g_2 = 2(4f_+ - 3)\, g_2 + 4g_3 \qquad (4.3)$$

In the same manner, the equations for the higher correlation functions can be calculated:

$$\frac{\partial}{\partial t} g_3 = 3(4f_+ - 3)\, g_3 - 6g_2^2 + 6g_4 \tag{4.4}$$

$$\frac{\partial}{\partial t} g_4 = 4(4f_+ - 3)\, g_4 - 8g_2 g_3 + 8g_5 \tag{4.5}$$

This regularity is no accident; we show by induction in Appendix A that the equation for the p-particle correlation function reads as follows:

$$\frac{\partial}{\partial t} g_p = p(4f_+ - 3)\, g_p - 2p(g_2\, g_{p-1} - g_{p+1}) \tag{4.6}$$

The p-particle correlation couples to f_+ and g_2 and also to g_{p-1} and g_{p+1}.

5. EXACT SOLUTION OF THE BOLTZMANN HIERARCHY

As shown by Spohn,[4,5] an exact solution of the Boltzmann hierarchy (3.2) is

$$f_p(e_1,..., e_p; t) = \int_0^1 \mu(df_1)_{\mathrm{B}} \prod_{i=1}^{p} [f_1(e_i; t)]_{\mathrm{B}} \tag{5.1}$$

as can be checked by insertion in Eq. (3.2). Here the subscript B indicates that the one-particle distribution functions $[f_1(e_i; t)]_{\mathrm{B}}$ on the right-hand side of (5.1) are the solutions of the Boltzmann equation (2.7). Thus, the exact reduced distribution functions may be interpreted as the moments of the various solutions of the Boltzmann equation. Their initial values are uniquely determined by the nonnegative probability measure $\mu(df_1)_{\mathrm{B}}$, which is defined on the space of the one-particle distribution functions. If molecular chaos is assumed initially, the probability measure is a δ-function, the reduced distribution function factorizes into a product of one-particle distribution functions, and Eq. (5.1) shows that this factorization holds for all times. This is known as the propagation of molecular chaos. On the other hand, if the correlations do not vanish initially, the width of the measure will represent the contribution of the various correlations, and the reduced distribution are statistical solutions of the Boltzmann equation. To make these statements more precise, we choose the probability measure as a rectangle distribution with a width $b-a$ $(0 \leqslant a < b \leqslant 1)$ and a height c. These parameters are connected through the normalization condition

$$\int_0^1 \mu(df_1) = 1$$

which yields here

$$c = 1/(b-a) \tag{5.2}$$

As shown in the previous section, we can restrict ourselves to the reduced distribution functions where all velocities are $+1$. In this case the initial value of the one-particle distribution function can be represented by a parameter x, defined in the interval $[0; 1]$. The mean value of the initial data of the p-particle distribution function is obtained from Eq. (5.1),

$$f_p(\{e_p\} = \{+1\}; t=0) = c\int_a^b x^p\,dx = \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)} = \langle x^p\rangle \tag{5.3}$$

while the dispersion

$$\sigma(t=0) = (\langle x^{2p}\rangle - \langle x^p\rangle^2)^{1/2}$$

represents the contribution of the correlations.

Since the solution of the Boltzmann equation (2.10) is known analytically [Eq. (2.11)], we can calculate the integral in (5.1) explicitly for the one-particle distribution

$$f_+(t) = \frac{1}{2} - \frac{e^{-t}}{2(1-e^{-t})} - c\frac{e^{-t}}{4(1-e^{-t})^2}\ln\frac{1-(2b-1)(1-e^{-t})}{1-(2a-1)(1-e^{-t})} \tag{5.4}$$

Note that we have chosen a special class of probability measures characterized only by two parameters, namely the mean value and the dispersion. While the initial values of all p-particle distribution functions f_p are uniquely determined by this probability measure, not all physically admissible initial values of the f_p with $p>2$ are obtained with a special measure. In practice, one would probably like to determine the measure for a given set of initial values of the reduced distribution functions. Here we want to compare the approximate solutions of the truncated hierarchy with the exact solution of the Boltzmann hierarchy (3.2). For that purpose it is more convenient to obtain the initial values from a given probability measure.

6. TRUNCATION OF THE HIERARCHY FOR THE CORRELATION FUNCTIONS

Although we have found an exact equation for the correlation functions (4.6), we cannot solve them to all orders without approximation, because of the coupling between g_p and g_{p+1}. In the following, we therefore close the hierarchy (4.6) by truncation at various levels, study the time evolution of the one-particle distribution function, and compare the results with the exact solution (5.4). The first equation (4.1) for f_+ has already been studied in Section 2 for the case $g_2 = 0$.

Let us now take the next step beyond that approximation and neglect g_3 in the second hierarchy equation (4.3):

$$\partial f_+/\partial t = (2f_+ - 1)(f_+ - 1) + 2g_2 \tag{6.1a}$$

$$\partial g_2/\partial t = 2(4f_+ - 3)\, g_2 \tag{6.1b}$$

We see at once that, if g_2 vanishes initially, it will stay at later times, the two equations decouple completely, and there remains the Boltzmann equation (2.10). For $g_2 \neq 0$, an important feature of Eqs. (6.1) is the existence of an additional fixed point. The equations

$$\begin{aligned} (2f_+ - 1)(f_+ - 1) + 2g_2 &= 0 \\ 2(4f_+ - 3)\, g_2 &= 0 \end{aligned} \tag{6.2}$$

are solved not only for the values

$$g_2 = 0, \qquad f_+ = 1, \qquad f_+ = \frac{1}{2} \tag{6.3}$$

already known, but also by the set

$$f_+ = \frac{3}{4} \qquad \text{and} \qquad g_2 = \frac{1}{16} \tag{6.4}$$

which is a saddle point, as shown in Appendix B. The coupled equations have been solved numerically.

The trajectories of the system are shown in Fig. 1 for various initial values in the f_+–g_2 plane. The existence of the saddle point means that there are initial values not leading to the equilibrium value $f_+ = \frac{1}{2}$, $g_2 = 0$, but to an increase of f_+ and g_2 toward infinity, which is unphysical.

We will now show that the admissible initial values for the correlation functions depend on the level of truncation. On the level of Eqs. (6.1), all values of f_+ and g_2 within the dotted curve of Fig. 1 allow a probability interpretation for the reduced two-particle distribution. This function becomes negative outside the dotted curve for some values of its arguments. At higher levels of truncation the area of admissible initial values becomes smaller as the probability interpretation also of the higher distribution functions has to be ensured. In the exact solution (5.1) all p-particle distribution functions are positive definite for all arguments. This allows us to

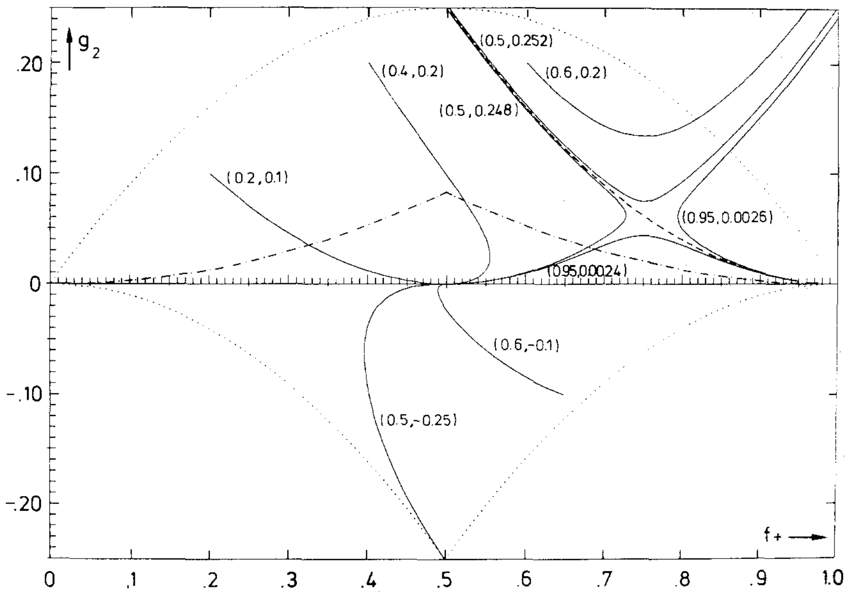

Fig. 1. The f_+–g_2 plane in the approximation where g_3 is neglected in the second hierarchy equation. There is an attractive fixed point at $(\frac{1}{2}, 0)$, a repulsive fixed point at $(1, 0)$, and a saddle point at $(\frac{3}{4}, \frac{1}{16})$. The solid curves labeled by the initial values $(f_+(t=0), g_2(t=0))$ are the trajectories of the system. The dashed curve separates the physical region below from the unphysical region above. Outside the area bounded by the dotted curve, the reduced two-particle distribution function becomes negative for some values of its arguments. The dashed-dotted curve peaked at $f_+ = 0.5$ bounds the physical initial values of the exact two-particle distribution.

calculate the admissible initial values for all correlation functions. For this purpose we use the cluster representation of the p-particle distribution function (3.7a) in the left-hand side of Eq. (5.3) and obtain for $t=0$

$$f_+^p + \sum_{i=1}^{p-2} \binom{p}{i} f_+^i g_{p-i} + g_p = \frac{b^{p+1} - (2f_+ - b)^{p+1}}{2(p+1)(b-f_+)} \tag{6.5}$$

where the relation

$$f_+(t=0) = \frac{1}{2}(a+b) \tag{6.6}$$

is used, which is obtained from Eq. (5.3) if $p=1$.

We set $p=2$ in Eq. (6.5) and obtain

$$g_2 = (b-f_+)^2/3 \tag{6.7}$$

In Eq. (6.7) the two-particle correlation becomes a maximum if the width parametrized by b is chosen as large as possible. Thus, we obtain for $f_+(t=0) \geq \frac{1}{2}$

$$g_2 = (1 - f_+)^2/3 \qquad (b = 1) \tag{6.8a}$$

and for $f_+(t=0) < \frac{1}{2}$

$$g_2 = f_+^2/3 \qquad (a = 0) \tag{6.8b}$$

where Eq. (6.6) is used to obtain Eq. (6.8b). Equations (6.8) are shown graphically by the dashed-dotted curve in Fig. 1. Thus, the area of initial values preserving a probability interpretation of the reduced two-particle distribution becomes considerably smaller if the higher distribution functions are also required to have a probability interpretation.

Let us now go beyond the approximation of Eqs. (6.1) and truncate the hierarchy at higher levels.

There the unphysical saddle point vanishes, as can be seen from Eq. (4.6), and all trajectories lead to the equilibrium point. It is remarkable that, if all correlations with the exception of g_2 vanish initially, they will be created in time, whereas if g_2 also vanishes at $t = 0$, no higher correlation

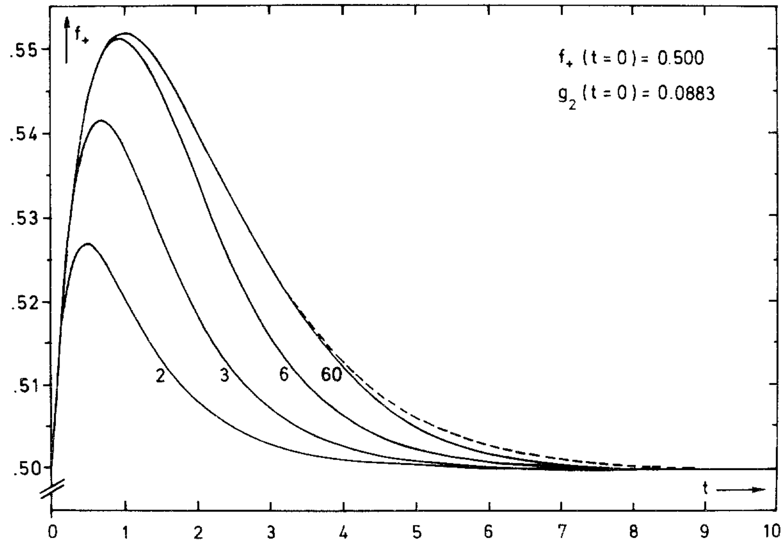

Fig. 2. Time evolution of f_+ with initial values lying in the peak of the dashed-dotted curve of Fig. 1. The solid curves are labeled according to the number of hierarchy equations taken into account. The dashed curve describes the exact solution corresponding to the initial values of the solid curves.

will be created at later times and the p-particle distribution remains a product of p one-particle distribution functions. This shows again the propagation of molecular chaos.

Now we can study the convergence of the truncation method by comparing with the exact solution in the McKean model. Figure 2 shows the time evolution of the exact one-particle distribution and its approximations obtained by truncating the hierarchy at increasing levels.

The initial values for the correlation functions are calculated from Eq. (6.5). The probability measure is chosen as a rectangular distribution with a width over the whole range of the interval [0; 1]. This guarantees a

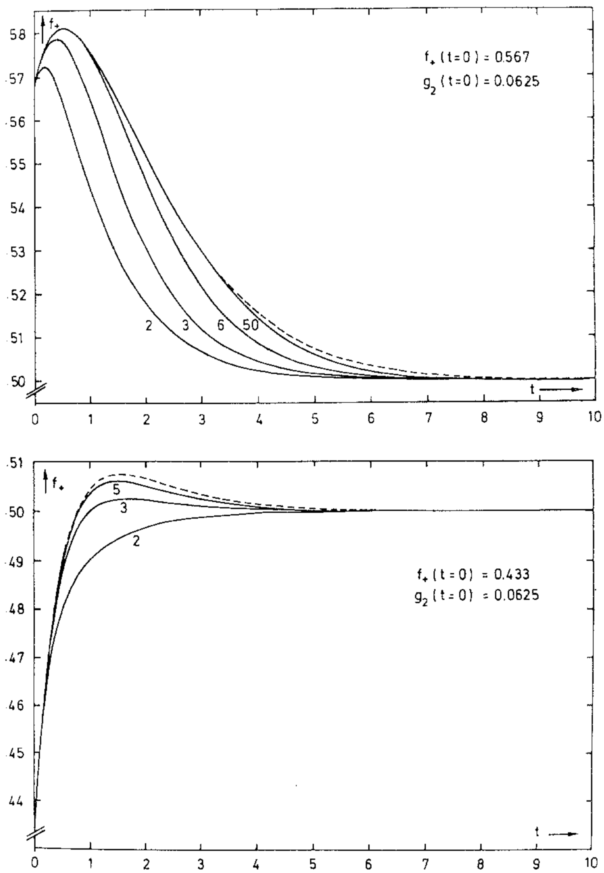

Fig. 3. Time evolution of f_+ with initial values lying on the right- and on the left-hand sides of the peak of the dashed-dotted curve of Fig. 1. The solid curves are labeled according to the number of hierarchy equations taken into account. The dashed curves describe the exact solution corresponding to the initial values of the solid curves.

maximum contribution of the correlations. The initial values of f_+ and g_2 corresponding to this measure are exactly on the peak of the dashed-dotted curve in Fig. 1. As shown in Fig. 2, the convergence of the approximations obtained by the truncation method toward the exact solution is rather bad.

One might query whether the convergence properties are influenced by the saddle point occurring in the approximation where g_3 is neglected in the second hierarchy equation. To investigate this question, we have chosen two sets of initial values. One set lies near the saddle point on the right branch of the dashed-dotted curve in Fig. 1, the other lies far from the saddle point as the left branch of this curve. The weight of correlations, i.e., the width of the rectangular distribution, is the same for both sets of initial values. It is shown in Fig. 3 that the convergence is much better for initial values lying far from the saddle point in the f_+–g_2 plane than for those lying near the saddle point. Thus, the saddle point has an influence on the convergence of the truncation method even in approximations in which it no longer exists.

7. DECAY OF THE p-PARTICLE CORRELATIONS

In the McKean model we are now able to confirm the assumption that the correlation functions decay faster than the one-particle distribution function tends to the equilibrium, i.e., they become increasingly less important for the description of a many-particle system near the equilibrium.[2]

We therefore rewrite Eq. (5.1) for the case where all particles have velocities $+1$ and insert the solution of the Boltzmann equation (2.11) in the right-hand side of Eq. (5.1). In addition, we use Eq. (3.7a) to calculate the time evolution of the correlation functions and obtain

$$\begin{aligned} f_+^p(t) + \sum_{i=1}^{p-2} \binom{p}{i} f_+^i(t)\, g_{p-i}(t) + g_p(t) \\ = \int_0^1 \mu(df_+(0)) \left[\frac{1}{2}\left(1 + \frac{Ae^{-t}}{1 - A(1-e^{-t})}\right)\right]^p \end{aligned} \tag{7.1}$$

Since we are interested in the vicinity of the equilibrium point, which is reached for $t \to \infty$, we linearize the left-hand side of Eq. (7.1) around this point and neglect the exponential term in the denominator of the right-hand side:

$$\begin{aligned} 2^{-p}[1 + 2p\varepsilon(t)] + \sum_{i=0}^{p-2} \binom{p}{i} 2^{-i} \eta_{p-i}(t) \\ = 2^{-p} \int_0^1 \mu(df_+(0)) \left\{1 + \frac{[2f_+(0) - 1]\, e^{-t}}{2[1 - f_+(0)]}\right\}^p \end{aligned} \tag{7.2}$$

where $\varepsilon(t)=f_+(t)-\frac{1}{2}$ and $\eta_p(t)=g_p(t)$, $|\varepsilon|$, $|\eta|\ll 1$. This formula can be brought into a more compact form:

$$2^{1-p}p\varepsilon+\sum_{i=0}^{p-2}\binom{p}{i}2^{-i}\eta_{p-i}=2^{-p}\sum_{i=0}^{p-1}\binom{p}{i}2^{-(p-i)}M_{p-i}e^{-(p-i)t} \quad (7.3)$$

where the abbrevation

$$M_{p-i}=\int_0^1 \mu(df_+(0))\left[\frac{2f_+(0)-1}{1-f_+(0)}\right]^{p-i}$$

has been used.

Equation (7.3) is solved by

$$\varepsilon(t)=\frac{1}{4}M_1e^{-t} \quad (7.4a)$$

$$\eta_p(t)=(M_p/2^{2p})\,e^{-pt} \quad (7.5a)$$

which can be verified by insertion.

Thus, we see that the p-particle correlations decay p times faster than the one-particle distribution tends to the equilibrium value. This can of course also be shown by truncation of Eq. (4.6) at the pth level and linearizing the whole system around the equilibrium point. In this way the solution of the set of coupled nonlinear differential equations is reduced to a linear eigenvalue problem which can be solved exactly.

8. CONCLUSIONS

We have shown that the McKean model is not only useful for a study of the one-particle properties of a many-particle system, but also allows an investigation of the p-particle correlations.

We transformed the BBGKY hierarchy of the McKean model into a simple system of coupled differential equations describing the time evolution of the p-particle correlation.

We determined the exact solution for the reduced distribution functions of the McKean model using the methods of Refs. 4–6 and compared the exact solution with the results of the truncation method at various levels.

Neglecting correlations of more than two particles leads to an unphysical saddle point, which vanishes again if more than two hierarchy equations are taken in account. The initial values that are acceptable for a physical interpretation of the various reduced distribution functions depend on the level at which the hierarchy is truncated. The coupled equations have been solved numerically up to $p = 60$ and the convergence of the one-particle distribution toward its exact shape has been studied. The effects of the higher correlations are particularly important in the vicinity of the unphysical saddle point in the $f_+ - g_2$ subspace because the convergence of such one-particle distribution functions is much worse than for those that evolve far from the saddle point.

For long times, the system approaches an equilibrium fixed point and the exact solution can be linearized showing the decay of the various correlations.

The p-particle correlations decay with relaxation times $\tau(g_p) = 1/p$ that are shorter than the relaxation time $\tau(f_+) = 1$ of the one-particle distribution function. Higher correlations thus become increasingly unimportant for an aged system. Our results confirm the conjecture of Kac[7] concerning the eigenvalues of the kinetic equation for the N-particle distribution function.

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APPENDIX A

To prove Eq. (4.6), we rewrite Eq. (3.8) in the form

$$\begin{aligned}
\frac{\partial}{\partial t}\left[f_+^p + \sum_{i=1}^{p-2}\binom{p}{i} f_+^i g_{p-i} + g_p\right]
&= p(2f_+^{p+1} - 3f_+^p + f_+^{p-1})\\
&\quad + \sum_{i=1}^{p-3} 2p\frac{p+1}{i+1}\binom{p}{i} f_+^{i+1} g_{p-i} - \sum_{i=1}^{p-2} 3p\binom{p}{i} f_+^i g_{p-i}\\
&\quad + \sum_{i=1}^{p-2} i\binom{p}{i} f_+^{i-1} g_{p-i} + p^2(p+1) f_+^{p-1} g_2 + 2p(p+1) f_+ g_p\\
&\quad - 3pg_p + 2pg_{p+1}
\end{aligned}$$

where all binominal coefficients have the form $\binom{p}{i}$. On the other hand, we can calculate the first two terms of the left side of (A.1),

$$\frac{\partial}{\partial t}\left[f_+^p + \sum_{i=1}^{p-2}\binom{p}{i} f_+^i g_{p-i}\right]$$
$$= pf_+^{p-1}\frac{\partial f_+}{\partial t} + \sum_{i=1}^{p-2}\binom{p}{i}\left(if_+^{i-1}\frac{\partial f_+}{\partial t} + f_+^i\frac{\partial g_{p-i}}{\partial t}\right) \tag{A.2}$$

with the help of (4.1) and with the induction assumption

$$\frac{\partial g_{p-i}}{\partial t} = (p-i)(4f_+ - 3)\, g_{p-i} - 2(p-i)\, g_{p-i-1}\, g_2 + 2(p-i)\, g_{p-i+1} \tag{A.3}$$

which is already valid for g_p, $p=2, 3, 4$ [see Eqs. (4.3)–(4.5)]. This leads to

$$\frac{\partial}{\partial t}\left[f_+^p + \sum_{i=1}^{p-2}\binom{p}{i} f_+^i g_{p-i}\right]$$
$$= p(2f_+^{p+1} - 3f_+^p + f_+^{p-1}) + p^2(p+1)\, f_+^{p-1} g_2$$
$$+ 2pg_{p-1}g_2 + \sum_{i=1}^{p-3} 2(2p-i)\binom{p}{i} f_+^{i+1} g_{p-i} - \sum_{i=1}^{p-2} 3p\binom{p}{i} f_+^i g_{p-i}$$
$$+ \sum_{i=1}^{p-2} i\binom{p}{i} f_+^{i-1} g_{p-i} + \sum_{i=1}^{p-3} \frac{2(p-i-1)(p-i)}{i+1}\binom{p}{i} f_+^{i+1} g_{p-i}$$
$$+ 2p(p-1)\, f_+ g_p \tag{A.4}$$

We obtain by subtraction of (A.4) from (A.1)

$$\frac{\partial}{\partial t} g_p = p(4f_+ - 3)\, g_p - 2pg_{p-1}\, g_2 + 2pg_{p+1} \tag{A.5}$$

which confirms the induction assumption.

APPENDIX B

We want to show by linearization Eqs. (6.1) around the various fixed points that

(i) $f_+ = 1$, $g_2 = 0$ is a repulsive fixed point

(ii) $f_+ = \frac{1}{2}$, $g_2 = 0$ is an attractive fixed point

(iii) $f_+ = \frac{3}{4}$, $g_2 = \frac{1}{16}$ is a saddle point

(i) $\varepsilon = f_+ - 1$, $g_2 = \eta$ ($|\varepsilon|, |\eta| \ll 1$): Insertion in (6.1) leads to

$$\begin{pmatrix}\dot{\varepsilon}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}1 & 2\\ 0 & 2\end{pmatrix}\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix} \tag{B.1}$$

The eigenvalues of this matrix are positive

$$\lambda_1 = 1, \qquad \lambda_2 = 2$$

which means that the solutions of (B.1) increase with the time and the fixed point can never be reached.

The complete solution of (B.1) is therefore

$$\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix} = C_1 e^{t}\begin{pmatrix}1\\ 0\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}2\\ 1\end{pmatrix} \tag{B.2}$$

(ii) $\varepsilon = f_+ - \frac{1}{2}$, $g_2 = \eta$: This leads to

$$\begin{pmatrix}\dot{\varepsilon}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}-1 & 2\\ 0 & -2\end{pmatrix}\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix}$$

with the eigenvalues

$$\lambda_1 = -1, \qquad \lambda_2 = -2$$

In contrast to (B.1), these eigenvalues are negative, leading to an exponential decay of ε and η, which means that the fixed point is reached for $t \to \infty$.

The solution is

$$\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix} = C_1 e^{-t}\begin{pmatrix}1\\ 0\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}2\\ -1\end{pmatrix}$$

and we obtain that the decay of η is much faster than that of ε.

(iii) $\varepsilon = f_+ - \frac{3}{4}$, $\eta = g_2 - \frac{1}{16}$: This leads to

$$\begin{pmatrix}\dot{\varepsilon}\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix}0 & 2\\ \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}\varepsilon\\ \eta\end{pmatrix}$$

with the eigenvalues

$$\lambda_1 = 1, \qquad \lambda_2 = -1$$

showing that this point is a saddle point.

From the solution

$$\begin{pmatrix} \varepsilon \\ \eta \end{pmatrix} = C_1 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

we obtain one direction in the $f_+ - g_2$ plane that is repulsive and one that is attractive (see Fig. 1).

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